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COMMENT

The potential for a homogeneous spheroid in a spheroidal coordinate system: I. At an exterior point

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Abstract. The explicit expressions of the external potential for a homogeneous spheroid in a rectangular coordinate system have been studied previously by several researchers. Presented here are the simplest forms of the potential in existing expressions, as derived from the reciprocal of the distance between two points expanded with the Legendre functions of the first and second kind in a spheroidal coordinate system. The numerical comparison of the potentials in spheroidal and rectangular coordinate systems shows exactly their identity. It is now feasible to calculate the gravitational potential for an astronomical body having spheroidal shape in its spheroidal coordinate system and to determine directly the equipotential surfaces by setting the corresponding spheroidal coordinate to a constant.

1. Introduction

The external potential for a homogeneous spheroid in a rectangular coordinate system was investigated by Kellogg (1929), Hopfner (1933) and MacMillan (1958). The expressions proposed by Kellogg are written in the form:

$$U_{\mathrm{e}}=\frac{6E}{f^2}\left[\frac{4x^2-2r^2-f^2}{2f}\ln\left(\frac{s-f}{s+f}\right)^{1/2}+\frac{s^2(2x^2-r^2)-2f^2x^2}{s(s^2-f^2)}\right] \qquad \text{for prolate spheroid} \tag{1a}$$

$$U_{\mathrm{e}}=\frac{6E}{f^2}\left(\frac{4z^2-2r^2+f^2}{2f}\sin^{-1}(f/s)+\frac{s^2(r^2-2z^2)-f^2r^2}{s^2(s^2-f^2)^{1/2}}\right) \qquad \text{for oblate spheroid} \tag{1b}$$

where E is the mass of the spheroid, f is the distance between the foci of a meridian section, s is the sum of the focal radii to the field point P, x or z is the distance from P to the equatorial plane, r is the distance from P to the rotational axis and U_{e} is the external potential for the spheroid. The formulae given by Hopfner are

$$\begin{aligned}V_{\mathrm{e}}=\pi ac^2\Bigg\{&\frac{-2}{(a^2-c^2)^{1/2}}\ln\left(\frac{(a^2+u)^{1/2}-l}{(a^2+u)^{1/2}+l}\right)^{1/2}\\&-\frac{(y^2+z^2)}{(a^2-c^2)^{3/2}}\left[\ln\left(\frac{(a^2+u)^{1/2}-l}{(a^2+u)^{1/2}+l}\right)^{1/2}+\frac{l(a^2+u)^{1/2}}{c^2+u}\right]\\&+\frac{2x^2}{(a^2-c^2)^{3/2}}\left[\ln\left(\frac{(a^2+u)^{1/2}-l}{(a^2+u)^{1/2}+l}\right)^{1/2}\right.\\&\left.\left.+\frac{l}{(a^2+u)^{1/2}}\right]\right\} \qquad \text{for prolate spheroid}\end{aligned} \tag{2a}$$

$$V_e = \pi a^2 c \left\{ \frac{2}{(a^2-c^2)^{1/2}} \tan^{-1}\left(\frac{a^2-c^2}{c^2+u}\right)^{1/2} - \frac{(x^2+y^2)}{(a^2-c^2)^{3/2}} \left[\tan^{-1}\left(\frac{a^2-c^2}{c^2+u}\right)^{1/2} - \frac{(a^2-c^2)^{1/2}(c^2+u)^{1/2}}{a^2+u} \right] + \frac{2z^2}{(a^2-c^2)^{3/2}} \left[\tan^{-1}\left(\frac{a^2-c^2}{c^2+u}\right)^{1/2} - \left(\frac{a^2-c^2}{c^2+u}\right)^{1/2} \right] \right\} \quad \text{for oblate spheroid} \tag{2b}$$

where l is the semifocal distance, a is the semimajor axis, c is the semiminor axis, u is a positive root of the equation for confocal ellipsoids:

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \tag{3}$$

in which $b = c$ for prolate and $a = b$ for oblate and V_e is the external potential. The forms derived by MacMillan are expressed as

$$V = \pi\sigma a^2 c \left(1 + \frac{x^2+y^2-2z^2}{2(c^2-a^2)}\right) \frac{2}{(c^2-a^2)^{1/2}} \sinh^{-1}\left(\frac{c^2-a^2}{a^2+k}\right)^{1/2} - \pi\sigma a^2 c \frac{(c^2+k)^{1/2}(x^2+y^2)}{(c^2-a^2)(a^2+k)} + \pi\sigma a^2 c \frac{2z^2}{(c^2-a^2)(c^2+k)^{1/2}} \quad \text{for prolate spheroid} \tag{4a}$$

$$V = \frac{2\pi\sigma a^2 c}{(a^2-c^2)^{1/2}} \left(1 - \frac{x^2+y^2-2z^2}{2(a^2-c^2)}\right) \sin^{-1}\left(\frac{a^2-c^2}{a^2+k}\right)^{1/2} + \pi\sigma a^2 c \frac{(c^2+k)^{1/2}(x^2+y^2)}{(a^2-c^2)(a^2+k)} - \pi\sigma a^2 c \frac{2z^2}{(a^2-c^2)(c^2+k)^{1/2}} \quad \text{for oblate spheroid} \tag{4b}$$

where σ is the mass density, c and a are the semimajor and semiminor axes for prolate, a and c are the semimajor and semiminor axes for oblate, k is the same as u in (2a) and (2b) and V is the external potential. In the former two sets of expressions, the x axis is the rotational axis for prolate, the z axis for oblate; in the latter set of expressions, the z axis is always the rotational axis for both.

Substituting in (1a) and (1b):

$$M \to E \tag{5a}$$

$$2l \to f \tag{5b}$$

$$(x^2+y^2) \to r^2 \tag{5c}$$

$$2(a^2+u)^{1/2} \to s \tag{5d}$$

in (2a) and (2b):

$$M \to \tfrac{4}{3}\pi a c^2 \quad \text{for prolate spheroid} \tag{6a}$$

$$M \to \tfrac{4}{3}\pi a^2 c \quad \text{for oblate spheroid} \tag{6b}$$

$$l \to (a^2-c^2)^{1/2} \quad \text{for both} \tag{6c}$$

and in (4*a*) and (4*b*):

$$M \to \tfrac{4}{3}\pi\sigma a^2 c \qquad \text{for both prolate and oblate spheroids} \tag{7a}$$

$$l \to (c^2 - a^2)^{1/2} \qquad \text{for prolate spheroid} \tag{7b}$$

$$l \to (a^2 - c^2)^{1/2} \qquad \text{for oblate spheroid} \tag{7c}$$

$$\ln\left(\frac{l + (c^2+k)^{1/2}}{(a^2+k)^{1/2}}\right) \to \sinh^{-1}\left(\frac{c^2-a^2}{a^2+k}\right)^{1/2} \qquad \text{for prolate spheroid} \tag{7d}$$

$$\tan^{-1}\frac{l}{(c^2+k)^{1/2}} \to \sin^{-1}\left(\frac{a^2-c^2}{a^2+k}\right)^{1/2} \qquad \text{for oblate spheroid} \tag{7e}$$

we can reconcile these three sets of expressions.

In many cases, it would be more convenient and intuitive to express the potential for a spheroid in a spheroidal coordinate system rather than in a rectangular coordinate system in order to determine the equipotential surfaces.

2. The spheroidal potential in a spheroidal coordinate system

The evaluation of the potential can usually be written in the integral form:

$$V = \iiint \mathrm{d}v/r \tag{8}$$

where r is the distance from the interior point of a given body to the field point and $\mathrm{d}v$ is the volume element for a spheroid:

$$\mathrm{d}v = l^3(\xi'^2 - \eta'^2)\,\mathrm{d}\xi'\,\mathrm{d}\eta'\,\mathrm{d}\phi' \tag{9}$$

in which l is the semifocal distance, ξ' is the spheroidal radial coordinate, η' is the spheroidal angular coordinate and ϕ' is the azimuthal angle.

The ratio of l to r in the spheroidal coordinate system can be expanded in terms of the Legendre function of the first and second kind (Hobson 1931):

$$\begin{aligned}\frac{l}{r} = &\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)P_n(\cos\theta')Q_n(\cosh\eta)P_n(\cosh\eta')\\ &+2\sum_{n=1}^{\infty}(2n+1)\sum_{m=1}^{n}(-1)^m\left(\frac{(n-m)!}{(n+m)!}\right)^2 P_n^m(\cos\theta)P_n^m(\cos\theta')\\ &\times Q_n^m(\cosh\eta)P_n^m(\cosh\eta')\cos m(\phi-\phi') \qquad \text{for } \eta > \eta'\end{aligned} \tag{10}$$

where the relations of coordinates between rectangular and spheroidal systems are

$$x = l\sinh\eta\sin\theta\cos\phi \tag{11a}$$

$$y = l\sinh\eta\sin\theta\sin\phi \tag{11b}$$

$$z = l\cosh\eta\cos\theta. \tag{11c}$$

By defining the new spheroidal coordinates ξ, η, ϕ instead of η, θ, ϕ in (11*a*), (11*b*) and (11*c*) in such a way that

$$x = l(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\phi \tag{12a}$$

$$y = l(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\phi \tag{12b}$$

$$z = l\xi\eta \tag{12c}$$

then (10) is converted to

$$\frac{l}{r}=\sum_{n=0}^{\infty}(2n+1)P_n(\eta)P_n(\eta')Q_n(\xi)P_n(\xi')+2\sum_{n=1}^{\infty}(2n+1)\sum_{m=1}^{n}(-1)^m\left(\frac{(n-m)!}{(n+m)!}\right)^2$$
$$\times P_n^m(\eta)P_n^m(\eta')Q_n^m(\xi)P_n^m(\xi')\cos m(\phi-\phi') \qquad \text{for } \xi>\xi'. \tag{13}$$

The substitution of (13) in (8) leads to

$$V=\frac{1}{l}\iiint\frac{l}{r}\,\mathrm{d}v=\frac{1}{l}\iiint\left(\sum_{n=0}^{\infty}(2n+1)P_n(\xi')Q_n(\xi)P_n(\eta')P_n(\eta)\right)$$
$$\times l^3(\xi'^2-\eta'^2)\,\mathrm{d}\xi'\,\mathrm{d}\eta'\,\mathrm{d}\phi'. \tag{14}$$

The integral of the second part in (13) over the spheroid vanishes because

$$\int_0^{2\pi}\cos[m(\phi-\phi')]\,\mathrm{d}\phi'\equiv 0 \qquad \text{for } m\neq 0. \tag{15}$$

The integral in (14) is composed of the following two parts:

$$V_1=2\pi l^2\sum_{n=0}^{\infty}Q_n(\xi)P_n(\eta)\int_{-1}^{+1}\int_1^{\xi_0}\xi'^2(2n+1)P_n(\xi')P_n(\eta')\,\mathrm{d}\xi'\,\mathrm{d}\eta' \tag{16a}$$

$$V_2=-2\pi l^2\sum_{n=0}^{\infty}Q_n(\xi)P_n(\eta)\int_{-1}^{+1}\int_1^{\xi_0}\eta'^2(2n+1)P_n(\xi')P_n(\eta')\,\mathrm{d}\xi'\,\mathrm{d}\eta' \tag{16b}$$

where ξ_0 is the value of the radial coordinate on the surface of the spheroid. With the help of the property of orthogonality for two associated Legendre functions:

$$\int_{-1}^{+1}P_n^m(x)P_{n'}^m(x)\,\mathrm{d}x=\frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}\delta_{nn'} \tag{17}$$

we can calculate the integrals in (16a) and (16b); after simplification procedures, they come out to be

$$V_1=\tfrac{4}{3}\pi l^2 Q_0(\xi)(\xi_0^3-1) \tag{18a}$$

$$V_2=-\tfrac{4}{3}\pi l^2 Q_2(\xi)P_2(\eta)\xi_0(\xi_0^2-1)-\tfrac{4}{3}\pi l^2 Q_0(\xi)(\xi_0-1). \tag{18b}$$

So we find that

$$V=V_1+V_2=[\tfrac{4}{3}\pi l^2\xi_0(\xi_0^2-1)](Q_0(\xi)-Q_2(\xi)P_2(\eta)). \tag{19}$$

The mass of the prolate spheroid is

$$M=\tfrac{4}{3}\pi ac^2=\tfrac{4}{3}\pi(l\xi_0)[l(\xi_0^2-1)^{1/2}]^2=\tfrac{4}{3}\pi l^3\xi_0(\xi_0^2-1) \tag{20}$$

where we had used the unit mass density in the integral of potential in (8). Thus, (19) becomes

$$V=(M/l)(Q_0(\xi)-Q_2(\xi)P_2(\eta)) \qquad \text{for prolate spheroid} \tag{21}$$

where the Legendre function of the second kind, $Q_0(\xi)$, is evaluated by

$$Q_0(\xi)=\tfrac{1}{2}\ln((\xi+1)/(\xi-1)) \tag{22a}$$

and ξ_0 is given by

$$\xi_0=a/(a^2-c^2)^{1/2}. \tag{22b}$$

For oblate spheroid, replacing ξ by $i\xi$ and l by $-il$ we can obtain

$$V=(iM/l)(Q_0(i\xi)-Q_2(i\xi)P_2(\eta)) \tag{23}$$

where

$$Q_0(i\xi)=\tfrac{1}{2}\ln((i\xi+1)/(i\xi-1))=i(\tan^{-1}\xi-\tfrac{1}{2}\pi) \tag{24a}$$

and ξ_0 is given by

$$\xi_0=c/(a^2-c^2)^{1/2}. \tag{24b}$$

3. Numerical results

The comparison between the expressions of potentials for prolate and oblate spheroids in spheroidal and rectangular coordinate systems can be performed by certain calculations.

For prolate spheroid, assume $l=25$ and $\xi_0=\frac{5}{3}$; suppose the spheroidal coordinates of the field point P to be

$$\xi=\tfrac{13}{5} \tag{25a}$$

$$\eta=0.6 \tag{25b}$$

then the corresponding rectangular coordinates will be

$$x=39 \tag{26a}$$

$$(y^2+z^2)^{1/2}=48. \tag{26b}$$

Therefore, the potentials computed in two different expressions are

$$V=0.016\,204\,710\,497\ldots \qquad \text{from (2}a\text{)} \tag{27a}$$

and

$$V=0.016\,204\,710\,497\ldots \qquad \text{from (21).} \tag{27b}$$

For oblate spheroid, assume $l=25$ and $\xi_0=\frac{4}{3}$; suppose the spheroidal coordinates of P to be

$$\xi=\tfrac{12}{5} \tag{28a}$$

$$\eta=0.6 \tag{28b}$$

i.e.

$$(x^2+y^2)^{1/2}=52 \tag{29a}$$

$$z=36. \tag{29b}$$

The potentials are

$$V=0.015\,805\,070\,122\ldots \qquad \text{from (2}b\text{)} \tag{30a}$$

and

$$V=0.015\,805\,070\,122\ldots \qquad \text{from (23).} \tag{30b}$$

The computational results of potentials for both prolate and oblate spheroids in their own systems are the same as those in rectangular coordinate systems.

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